Backward error for continued fractions

Irene Claudia Noharinaivo (irenen@aims.ac.za)
African Institute for Mathematical Sciences (AIMS)

Supervised by: Prof. Robert Corless and Prof. Nicolas Fillion
Western University, Ontario, Canada
Simon Fraser University, Canada

23 May 2019

Submitted in partial fulfillment of a structured masters degree at AIMS South Africa

Abstract

The synthesis between continued fractions and numerical analysis is explored in the paper "Chaos and Continued fractions" (Corless, 1992). The underlying idea is that the numerical simulation of the *Gauss map* generates continued fractions. In dynamical systems context, this article also investigates the conditions under which orbits of the *Gauss map* are periodic as well as the consequences of working with floating point numbers. This essay project analyzes this research and then explore the dynamics of the *Gauss map* in half-precision floating point.

Declaration

I, the undersigned, hereby declare that the work contained in this research project is my original work, and that any work done by others or by myself previously has been acknowledged and referenced accordingly.

Irene Claudia Noharinaivo, 23 May 2019

Contents

1. Introduction and preliminaries

Dynamical system is the study of time evolution of points in a specified geometrical space. It appears in many areas of study which require mathematical modelling. For instance the study of long-term behavior of certain species population in biology. The evolution function takes several points depending on the system but our main focus will be on the study of one dimensional dynamical system which is the *Gauss map*. The connection between this map and continued fractions gives an interesting result in dynamical interpretation. The aim of this essay is to understand the dynamics of the *Gauss map*. The Chapter 2 will evaluate the study of orbits in general, whilst in the Chapter 3, we specialize to a system within the framework of fixed precision floating point arithmetic.

1.1 Continued fractions

The term *continued fractions* was first used by **John Wallis** in 1653 in his book *Arithmetica* (Widž, 2009). In general, continued fractions are used to give the best possible rational approximations to irrational numbers. A so-called "simple continued fraction" of a number γ has an expression of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ldots}}}$$

where the first term a_0 is an integer that could be zero and the other a_i's are positive integers. They are called *elements* or *partial quotients* of the continued fraction. By convention, γ can also be written as $\gamma = [a_0; a_1, a_2, \ldots]$.

One way to represent a rational number p/q where $0 \neq q$ and $p \in \mathbb{Z}$ in the form of a finite simple continued fraction is by using Euclid's algorithm for computing the greatest common divisor. The quotients that arise in the Euclidean algorithm are precisely the partial quotients. The last nonzero remainder, which is in fact the greatest common divisor, appears as the numerator of last fractional part 2.1.1 of the continued fraction.

As an example, Euclid's algorithm applied to $\frac{p}{q} = \frac{97}{38}$ for computing the $\gcd$ is:

$$\begin{aligned} 97 &= 38 \times \underline{2} + 21 \\ 38 &= 21 \times \underline{1} + 17 \\ 21 &= 17 \times \underline{1} + 4 \\ 17 &= 4 \times \underline{4} + 1 \\ 4 &= 1 \times \underline{4} + 0. \end{aligned}$$

This yields the following expansion of $\frac{p}{q}$ in the form of a continued fraction:

$$\frac{p}{q} = 2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{4}}}} = [2; 1, 1, 4, 4].$$

Furthermore, since Euclid's algorithm terminates in $O(\log(\min(p,q)))$ operation, we deduce that any rational number can be represented as a finite simple continued fraction. Conversely, any finite simple continued fraction represents a rational number.

Since the expansion of rational numbers into continued fractions is straightforward, we turn our attention to the continued fraction expansion of irrational numbers. Most authorities agree that the modern theory of continued fraction began with the writings of **Rafael Bombelli**[1] in his treatise on algebra (Widž, 2009). He introduced square roots and essentially indicates that

$$\sqrt{a^2+b} = a + \cfrac{b}{2a + \cfrac{b}{2a + \dots}}$$

The convergence of this expansion was only proved later.

Later mathematicians working with continued fractions include **Pietro Antonio Cataldi**[2] and **Lord Brouncker**[3] (Widž, 2009). The typical method to obtain the continued fraction representation of a number γ especially for irrational numbers can be interpreted by the following algorithm:

First notice that γ can be written as the sum of its integer part, denoted n_0, and its fractional part denoted γ_0 (if it is not zero). Then we invert γ_0 and proceed identically, and n_1, γ_1 play the role of n_0 and γ_0 respectively. Again we iterate it a while, a time which we will define later on. But if γ is a rational number, this time will be finite. For example of an irrational γ, the continued fraction representation of $\sqrt{6}$ is:

$$\sqrt{6} = 2 + \cfrac{1}{2 + \cfrac{1}{4 + \cfrac{1}{2 + \dots}}}$$

Proof. Since $2^2 < 6 < 3^2$, then $\sqrt{6} = 2 + \alpha$ with $\alpha \in (0,1)$. The first step is to invert $\alpha = \sqrt{6} - 2$. We have:

$$\frac{1}{\alpha} = \frac{\sqrt{6}+2}{6-4} = \frac{\sqrt{6}+2}{2} = \frac{4+\sqrt{6}-2}{2} = 2 + \frac{\alpha}{2} \quad \text{where} \quad \frac{\alpha}{2} \in (0,1).$$

Thus $\sqrt{6} = 2 + \frac{1}{\frac{1}{\alpha}} = 2 + \frac{1}{2+\frac{\alpha}{2}}$, and we invert again $\alpha/2$. So we have:

$$\frac{1}{\frac{\alpha}{2}} = \frac{2}{\sqrt{6}-2} = \frac{2(\sqrt{6}+2)}{6-4} = 2 + \sqrt{6} = 4 + \alpha.$$

Finally,

$$\sqrt{6} = 2 + \frac{1}{2 + \frac{1}{4+\alpha}} \quad \text{at which the next step comes back to the first one.}$$

Therefore $\sqrt{6} = [2; 2, 4, 2, 4, 2, 4, \dots]$. □

[1] Rafael Bombelli (1530-1572) is a native of Bologna. He wrote the book entitled "Algebra" which gives a thorough account of the algebra and his contribution to complex numbers. He planned to publish five volumes of this book but unfortunately he was never able to publish the last two since he died after the publication of the first three.

[2] Pietro Antonio Cataldi (1548-1626) is a native Bologna. In 1613, he published "*Trattato del modo brevissimo di trovar la radice quadra delli numeri*" where he showed that the square root of a number is found through the use of infinite series and unlimited continued fractions.

[3] Lord Brouncker (1620-1684), the first President of the Royal Society. He transformed the interesting infinite product $\frac{4}{\pi}$ discovered by John Wallis into a continued fraction expansion.

1.2 Half precision floating points

Floating point numbers are numeric values with zero or non-zero fractional parts. They are an important data type in computation which is now being used extensively in high-speed applications. All floating point values are represented with a normalized scientific notation defined in the following example:

$$67.21654 = 0.6721654 \times 10^2.$$

The first part is called the *Mantissa*; or significand, and the second part is the *exponent* in the decimal format. In a computer, these numbers have a specific number of bits allocated for storage for both the mantissa and the exponent. The limit of the magnitude or range of the floating point numbers stands for the finite number of bits in the exponent. In case of the finite number of the mantissa bits, it bounds the number of significant digits or the precision of the floating point numbers. Thus the main difference between arbitrary real numbers and floating point numbers is that real numbers are infinite in terms of range and precision whereas floating point numbers are finite in both cases (Konsor, August 15, 2012).

The standard used in most of computer hardware to store and process the floating point numbers is developed by the Institute of of Electrical and Electronic Engineers (IEEE 754) in 1985 and enhanced in 2008. In this case, instead of using the decimal format, all floating point values use a normalized binary format. The representation of a binary floating point number is as follows:

$$(\text{sign})\times 1.\text{mantissa} \times 2^{\pm\text{exponent}},$$

where the sign is one bit (by convention, 0 for positive and 1 for negative), the mantissa is a binary fraction with a hidden non-zero leading bit, and the exponent is a binary integer. The most commonly used level of precision is the single precision with width 32 bits and the double precision with width 64 bits. As part of this essay we will not be focusing on these two but rather on the half precision floating point numbers because this is increasingly being used for speed (Konsor, August 15, 2012).

Half precision floats are based on 16-bit floating-point numbers, known as the *binary 16* or half-floats in IEEE 754-2008 standard, the half size of the 32-bit single precision floats. They have lower precision and smaller range and must be converted to/from 32-bit floats before they are operated on. The main reason for using them is that half-floats use less memory. As with all floating-point numbers, they have relatively high precision for floating-point values near zero, but have low precision for numbers far from zero (Konsor, August 15, 2012). The representation of the binary 16 has the following format:

- sign : 1 bit
- exponent : 5 bits
- mantissa : 11 bits (10 explicitly stored).

The commonly used binary format is arranged in this form:

sign-exponent-mantissa

For example 0 00001 00010101111.

However in terms of exponent encoding, the half precision floats use an offset binary known as the exponent bias in the IEEE 754 standard. Thus in order to get the right value of the exponent, we have to substract the offset 15 by the stored exponent.

Normalized numbers are numbers written in scientific notation in a way that there is only one nonzero integer part. In our case, this integer part is the number one. Any nonzero number with magnitude smaller than the smallest normal number is called a subnormal number. Infinities (Inf) and Not a Number (NaN) are special values used as replacement values when there is an overflow or an invalid operation. For precise definition of each different numbers in binary16, the chart below shows how to convert the binary16 into decimal. e stands for the exponent, s the signbit and m the mantissabit.

	Normal numbers	Subnormal numbers	0 and -0	$\pm$ Infinity	NaN
Exponent	00001,...,11110	00000	00000	11111	11111
Mantissa	zero or nonzero	nonzero	zero	zero	nonzero
Equation	$(-1)^{\mathsf{s}} \times 2^{\mathsf{e}\text{-}15} \times 1.\mathsf{m}_2$	$(-1)^{\mathsf{s}} \times 2^{-14} \times 0.\mathsf{m}_2$	$(-1)^{\mathsf{s}} \times 2^{-14} \times 0.\mathsf{m}_2$		

As an example, let us convert the two binary16 numbers 0 00000 1111111111 and 0 00001 0000000000 into decimal format. According the formula, we have

$$0 \quad 00000 \quad 1111111111 = (-1)^0 \times 2^{-14} \times 0.1111111111_2$$
$$0 \quad 00001 \quad 0000000000 = (-1)^0 \times 2^{1-15} \times 1.0000000000_2$$

Thus we have to convert this decimal binary into decimal.

$$\begin{aligned} 0.1111111111_2 &= \sum_{i=1}^{10} \left(\frac{1}{2}\right)^i \\ &= \frac{1-(\frac{1}{2})^{11}}{1-(\frac{1}{2})^1} - 1 = \frac{1023}{1024} \\ 1.0000000000_2 &= 1 + 0 = 1 \end{aligned}$$

Therefore

$$0 \quad 00000 \quad 1111111111 = \frac{1}{16384} \times \frac{1023}{1024} \approx 6.0976 \times 10^{-5}$$
$$0 \quad 00001 \quad 0000000000 = \frac{1}{16384} \approx 6.1035 \times 10^{-5}$$

The first example illustrates the largest subnormal number and the second one represents the smallest positive normal number.

2. The Gauss map as a chaotic discrete dynamical system

In this Chapter, we define and study several properties of the *Gauss map* and then relate it to continued fractions.

2.1 The Gauss map and continued fractions

2.1.1 Definition. Let x be a non-negative real number. The fractional part of x, denoted as $x \mod 1$, is given by:

$$x \mod 1 = x - [x]$$

where $[x]$ is the integer part of x.

2.1.2 Definition. The *Gauss map* $G : [0,1] \to [0,1]$ is the following map:

$$G(x) = \begin{cases} 0 & \text{if} \quad x = 0 \\ \frac{1}{x} \mod 1 & \text{if} \quad 0 < x \leq 1. \end{cases}$$

Notice that

$$\left[\frac{1}{x}\right] = n \iff n \leq \frac{1}{x} < n+1 \iff \frac{1}{n+1} < x \leq \frac{1}{n}.$$

Then the *Gauss map* can be expressed as:

$$G(x) = \begin{cases} 0 & \text{if} \quad x = 0 \\ \frac{1}{x} - n & \text{if} \quad \frac{1}{n+1} < x \leq \frac{1}{n} \end{cases} \quad \text{for} \quad n \in \mathbb{N}^*$$

where its graph is described below.

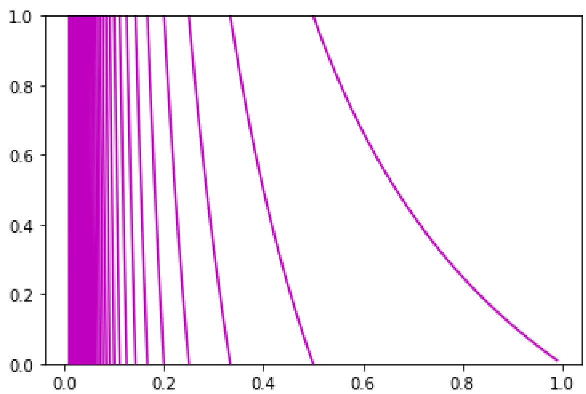

Figure 2.1: Gauss map

The restriction of $G : (\frac{1}{n+1}, \frac{1}{n}] \to [0,1)$ is monotone, surjective and invertible as we see on the picture. Notice that at values $x = \frac{1}{n}$, for all n nonzero positive integer, there are an infinite number of discontinuities. Because of these breaks, this function is not continuous. However the important property that we are interested in is not caused by the discontinuity. If we view G as a map of the circle onto the circle, so that we put together the ends of the interval, all these jump discontinuities will be removed. The picture below shows the visualization of the *Gauss map* lying in the torus.

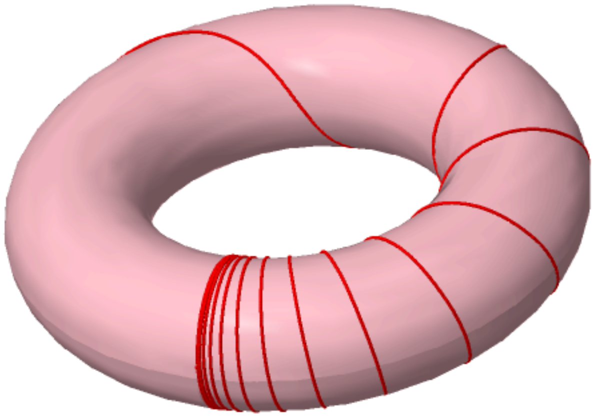

Figure 2.2: Graph of the *Gauss map* lying in the torus.

2.1.3 Definition. A discrete dynamical system is a map $f : X \to X$ where X is a system with $x \in X$ as an initial condition. This system changes as time goes on and may appear discretely or continuously.

From the algorithm stated in Chapter 1.1 to compute the continued fraction, we can define the following iteration:

$$\begin{cases} \gamma_{k+1} = \frac{1}{\gamma_k} \mod 1 = G(\gamma_k) \\ n_{k+1} = \left[\frac{1}{\gamma_k}\right] \end{cases} \qquad \text{for} \quad k \in \mathbb{N}$$

with γ_0 as an initial condition. It follows that the *Gauss map* is a discrete dynamical system.

2.1.4 Definition. We call the sequence $(G^k(x_0))_{k\geq 0}$ the orbit of the initial point x_0 under the *Gauss map* and denoted by $orb(x_0)$.

Studying the orbits in a dynamical system reveals important features about the time evolution of the system. In the next Section, we will take a closer look at orbits under the *Gauss map*.

2.1.5 Theorem. *Let $x \in [0, 1]$ and $n_1, n_2, ...n_k, ...$ the integers part that arise from the computation of $orb(x)$ under G. Then $x = [n_1, ..., n_k, ...]$.*

Proof. Let us remark first that if

$$n_1 = \left[\frac{1}{x}\right], \quad \text{then} \quad G(x) = \frac{1}{x} - n_1. \quad \text{Hence} \quad x = \frac{1}{n_1 + G(x)}.$$

Repeating the procedure with $G(x)$, we obtain

$$n_2 = \left[\frac{1}{G(x)}\right] \Longrightarrow G(x) = \frac{1}{n_2 + G^2(x)}$$

where $G^2(x) = G(G(x))$.

Now let us prove the theorem by induction. Suppose that

$$n_k = \left[\frac{1}{G^{k-1}(x)}\right] \quad \text{and} \quad x = \cfrac{1}{n_1 + \cfrac{1}{n_2 + ... \cfrac{1}{n_k + G^k(x)}}}$$

We note that this is already true for $n = 1$. Suppose that it is true for any k and consider $k + 1$.

We have $n_{k+1} = \left[\frac{1}{G^k(x)}\right]$ and by definition of G,

$$G^{k+1}(x) = \frac{1}{G^k(x)} - \left[\frac{1}{G^k(x)}\right] = \frac{1}{G^k(x)} - n_{k+1} \iff G^k(x) = \frac{1}{n_{k+1} + G^{k+1}(x)}.$$

□

2.1.6 Remark. The theorem holds in case $x \in \mathbb{Q}$. There exists an i such that $G^i(x) = 0$, and hence for all $j \geq i$, $G^j(x) = 0$. Thus the entries of the continued fraction expansion of x are given by the finite integer parts of the iteration of $orb(x)$ under G.

We deduce that continued fractions can be generated by the *Gauss map*. Thus, from now, there will be many implications for the dynamics of the *Gauss map* while operating the continued fractions.

2.1.7 Theorem. *If* $x = [n_1, n_2, ...]$*, then* $G^{(m)}(x) = [n_m, n_{m+1}, ...]$ *for all* $m \geq 0$*.*

Proof. We proceed by induction. We define $G^{(0)}(x) = x$. Hence for $m = 0$, the statement is verified. Suppose that it is true for all $m = k$, and let us demonstrate that $G^{(k+1)}(x) = [n_{k+1}, n_{k+2}, ...]$.

By the inductive hypothesis, we have

$$\begin{aligned} G^{(k)}(x) &= [n_k, n_{k+1}, ...] \\ &= \cfrac{1}{n_k + \cfrac{1}{n_{k+1} + ...}}. \end{aligned}$$

By inverting,

$$\frac{1}{G^{(k)}(x)} = n_k + \cfrac{1}{n_{k+1} + \cfrac{1}{n_{k+2} + ...}},$$

and by taking the fractional part, we obtain

$$\left\{\frac{1}{G^{(k)}(x)}\right\} = G^{(k+1)}(x) = \cfrac{1}{n_{k+1} + \cfrac{1}{n_{k+2} + ...}}.$$

□

This theorem shows that the *Gauss map* moves the partial quotients of the continued fraction expansion by an one-sided shift.

2.2 Periodic and fixed points

Now we will focus more on the *Gauss map* as part of a discrete dynamical system, but first we will introduce some fundamental notions in the setting of one-dimensional dynamics. Orbits play an important role in dynamics since they predict the behavior of the system. The following definitions are some of its properties.

2.2.1 Definition. The α*-limit set* of $orb(x_0)$ is the set of all initial points having orbits approaching $orb(x_0)$ as time goes to infinity. The ω*-limit set* of $orb(x_0)$ is the set of its accumulation points.

2.2.2 Definition. Any point x which satisfies $G(x) = x$ is called a fixed point. The point x_0 is a periodic point m of G if $G^m(x_0) = (x_0)$ but $G^i(x_0) \neq 0$ for $0 < i < m$. The least number m which verifies this property is the period of $orb(x_0)$.

2.2.3 Definition. Let x_0 be a periodic point such that $|G^{(n)}(x_0)| < 1$ for $n \geq 1$. Then x_0 is called a periodic point attractor of period n. If this is not the case, then x_0 is called a repelling periodic point of period n.

2.2.4 Example. From the figure 2.1, if we draw the line $y = x$ as seen in 2.3, the first two fixed points are given by the positive solution in $[0, 1]$ of the following equations:

$$x_1 = \frac{1}{x_1} - 1 \quad \text{and} \quad x_2 = \frac{1}{x_2} - 2$$

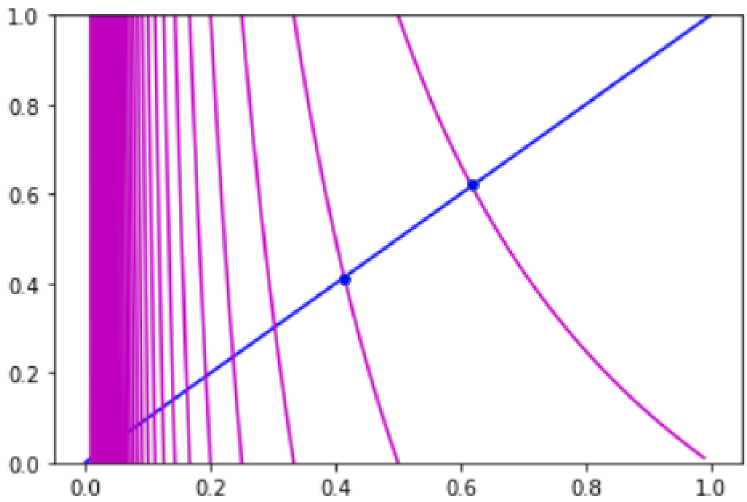

Figure 2.3: Fixed points of the Gauss map

Thus we have

$$\begin{aligned} x_1 = \frac{1}{x_1} - 1 &\Longrightarrow x_1^2 + x_1 - 1 = 0 \\ &\Longrightarrow x_1 = \frac{\sqrt{5} - 1}{2}, \end{aligned}$$

which is called the *Golden mean*. Since $x_1 = \frac{1}{x_1} - 1$, then $x_1 = \frac{1}{1+x_1}$. It follows that its continued fraction representation is

$$x_1 = \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+...}}}$$

denoted by $[1, 1, 1, ...]$.

For the second fixed point, we have

$$\begin{aligned} x_2 = \frac{1}{x_2} - 2 &\Longrightarrow x_2^2 + 2x_2 - 1 = 0 \\ &\Longrightarrow x_2 = \sqrt{2} - 1, \end{aligned}$$

which is called the *Silver mean* and proceeding the same way, its continued fraction representation is

$$x_2 = \cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+...}}}$$

denoted by $[2, 2, 2, ...]$.

Thus 1 and 2 are periodic points of the *Gauss map*.

Further, there are some classical theorems to study the periodic points and the fixed points. They are stated as follows.

2.2.5 Definition. A root of a quadratic equation with integer coefficients is called a reduced quadratic irrational.

2.2.6 Theorem. *(Galois) Let γ be a reduced quadratic irrational such that $\gamma > 1$ and its conjugate lies in the interval $(-1, 0)$. Then its continued fraction is purely periodic. The reciprocal is also true.*

Proof. The proof can be seen in (Olds, 1963). □

2.2.7 Corollary. (Corless et al., 1990) The reciprocals of the reduced quadratic irrational numbers give the periodic points of the *Gauss map*. Furthermore, they are dense in $[0, 1)$.

Proof. Let $\gamma = n_0 + [n_1, n_2, ...]$ be a reduced quadratic irrational. From Theorem 2.2.6, γ is purely periodic. Now recall that the *Gauss map* is given by the shift property, so as long as the continued fraction of γ is periodic then $G(\gamma)$ will also be periodic. Thus the periodic points are this reciprocal. As far as the density is concerned, the continued fraction expansion of a periodic point may initiate in the same way as any given continued fraction so it will ultimately be close to any given number. □

2.2.8 Example. Consider $\tau = \frac{\sqrt{5}+1}{2}$, the *Golden ratio*, which satisfies the quadratic equation $\tau^2 - \tau - 1 = 0$. Its continued fraction expansion is given by $\tau = 1 + [1, 1, 1, ...]$. The other root of this quadratic is $\frac{-1}{\tau} = \frac{1-\sqrt{5}}{2}$ which lies in the interval $(-1, 0)$. According to the corollary 2.2.7, the reciprocal of the reduced quadratic irrational $\frac{1}{\tau}$ is the periodic point and we have seen in the example 2.2.4 that it is the *Golden mean* with period 1.

Sometimes, this periodicity of the *Gauss map* may occur infinitely meaning that each period has many points. For instance, $[n_1, n_2, ..., n_k, n_1, n_2, ...]$ has period k for any integers $n_1, n_2, ..., n_k$.

2.2.9 Definition. A map f is said to be sensitive to initial conditions if for any pair of points which in the beginning are arbitrary close, have orbits that move apart at an exponential rate.

2.2.10 Definition. (Devaney, 1989) A map f is said to be *chaotic* on any given interval if:

1. f is sensitive to initial conditions.
2. f is topologically transitive.
3. its periodic points are dense in the interval.

Topological transitivity is dependent on the density of the orbits. For more details, one can consult (Sergiy Kolyada). In our case, we restrict our study to the sensitivity to initial conditions and the periodicity of orbits. These are the main characteristics of a chaotic system, including that chaotic maps possess a degree of unpredictability. Arbitrarily small changes to the initial points for orbits may lead significant errors throughout its future behavior.The following theorem emphasizes the sensitivity of the Gauss map.

2.2.11 Theorem. *(Lagrange) Let* $\gamma = [a_1, a_2, ..., a_i, n_1, n_2, ..., n_k, n_1, n_2, ..., n_k, ...]$ *be an ultimately periodic continued fraction with transients* $a_1, ..., a_i$ *at the beginning of a periodic continued fraction. Then* γ *is a quadratic irrational and the reciprocal is also true.*

Proof. See Olds (Olds, 1963). □

2.2.12 Corollary. (Corless et al., 1990) The *Gauss map* is sensitive to initial conditions.

Proof. Recall that an attractor is a point which attracts orbits. We have seen that the fixed point 0 attracts all rational initial points while a periodic orbit attracts quadratic irrational numbers. These two sets are dense in the interval $[0, 1)$. Now let us check the rate of separation of orbits. Consider all points in a small interval I of width ε. According to the pigeonhole principle [1], a rational number of the form $p/n = [a_1, a_2, ..., a_i]$ must be contained in this interval such that n is the smallest integer larger than $1/\varepsilon$. Since $O(\log(n))$ is the number of iterations of the *Gauss map* needed to reach zero, it will be $O(\log(\varepsilon))$ for this initial point. Since the quadratic irrationals are dense, the interval I contains a purely periodic point. Thus for a large enough N, $[a_1, a_2, ..., a_i, N, 1, 1, 1, ...]$ is the continued fraction expansion of a point in I. Thus, the orbit under G starting from this point ends up on the fixed point at $1/\tau$. Then, the separation is in an exponential rate. □

2.2.13 Remark. The continued fraction expansions for non-quadratic irrationals also exist. They are seen to have initial points whose orbits are not periodic under the Gauss map.

As a result, the *Gauss map* follows the requirements being a chaotic discrete dynamical system.

2.3 Lyapunov exponents

Numerically, a Lyapunov exponent is a quantitative measurement of a chaotic dynamical system. The *Gauss map* has already been seen to be sensitive to initial conditions, meaning that the orbits move away from each other over time. However, we do not know exactly the rate value at which these orbits become separated, but we have assumed that the trajectories are separating exponentially.

[1] In general, the pigeonhole principle states that taking n items and putting them in a m containers where $n > m$, there will at least one container which contains more than one item.

2.3.1 Definition. For two orbits under the *Gauss map* in an ε neighborhood, we have

$$|G^n(x+\varepsilon) - G^n(x)| \approx \varepsilon e^{n\lambda}$$

where λ is called the Lyapunov exponent.

2.3.2 Remark. Linear divergence appears if λ as defined above is equal to zero. If it is positive, then the system is sensitive to initial conditions. Otherwise if it is negative, the trajectories of orbits will decrease in time .

2.3.3 Definition. The *Lyapunov exponent* of the *Gauss map* at the initial point γ_0 is defined by

$$\lambda(\gamma_0) = \lim_{n\to\infty} \frac{1}{n} \ln \left(\prod_{i=1}^{n} |G'(\gamma_i)| \right)$$

whenever this limit exists. It is clear that if the initial point γ_0 is a rational, the limit does not exist.

This limit gives the average value of this function logarithm. From the Figure 2.1, the slope looks like it is getting steeper and steeper and it is not clear whether this is finite or not. However, the region in which it is happening becomes smaller, so there is a possibility that this will actually be finite. It turns out that the *Lyapunov exponent* exists and is given by:

$$\lambda(\gamma_0) = \frac{\pi^2}{6\ln 2}.$$

The explicit calculation of this will be shown in the next Section using the *ergodic theory*.

2.3.4 Definition. For a fixed point $\alpha_N = [N, N, N, ...]$ he *Lyapunov exponents* is as follows:

$$\begin{aligned}
\lambda(\alpha_N) &= \lim_{n\to\infty} \frac{1}{n} \ln \left(\prod_{i=1}^{n} |G'(\gamma_i)| \right) \\
&= \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n} \ln |G'(\alpha_N)| \\
&= \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n} \ln | - \frac{1}{\alpha_N^2}| \\
&= -2 \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n} \ln(\alpha_N) \\
&= 2\ln(\frac{1}{\alpha_N})
\end{aligned}$$

2.3.5 Example. We have already seen that the *Golden mean* $\frac{1}{\tau}$ is periodic with period 1 so its *Lyapunov exponent* is

$$\lambda(\frac{1}{\tau}) = 2\ln\tau \approx 0.96..$$

This example points out the fact that the *Gauss map* is chaotic since we have seen one point which has positive *Lyapunov exponent*. The following result shows that this value is the smallest *Lyapunov exponent* under G.

2.3.6 Theorem. *There are no orbits under the Gauss map which have Lyapunov exponents smaller than* $\lambda(\frac{1}{\tau}) = 2\ln\tau$.

Proof. Let $\gamma = [n_1, n_2, n_3, ...]$ be any initial point in $(0, 1)$ such that

$$\lambda(\gamma) = \lim_{N\to\infty} \frac{1}{N} \ln \left(\prod_{i=1}^{N} |G'(\gamma_i)| \right)$$

exists. The aim is to show that the product $\prod_{i=0}^{N}(1/\gamma_i^2)$ must be at least τ^{2N}. Consider the sequence γ_k and γ_{k+1} of the orbit γ related by

$$\gamma_k = \frac{1}{n_{k+1} + \gamma_{k+1}}.$$

Note that if $k = N$ then it increases the product by one term. Now we have two cases.

If $\gamma_k \leq \frac{1}{\tau}$, then $\tau^2 \leq \frac{1}{\gamma_k^2}$ which means that the input of $\frac{1}{\tau}$ to product is at least τ^2.

If $\gamma_k > \frac{1}{\tau}$, then

$$\begin{aligned}\gamma_k\gamma_{k+1} &= \frac{\gamma_{k+1}}{n_{k+1} + \gamma_{k+1}} = 1 - n_{k+1}\gamma_k \\ &\leq 1 - \gamma_k < 1 - \frac{1}{\tau} = \frac{1}{\tau^2}\end{aligned}$$

Thus the contribution of $\frac{1}{\gamma_k^2\gamma_{k+1}^2}$ to the product is at least τ^4. So, the theorem is proved. □

2.4 Ergodicity behavior

Ergodic theory is the study of the long term average behavior of systems envolving in time. Indeed, we would like to study the orbits of the *Gauss map* and how they change as time goes on. To illustrate this, these transformations preserve the structure of measure space in the study of this theory (Billingsley, 1965). The measure used has a relation to probability but we are not focusing too much on this.

2.4.1 Lemma. The *Gauss map* preserves the *Borel measure*[2], so-called μ in $[0, 1]$ given by

$$\mu(A) := \int_A \frac{1}{1+x} dx$$

for any Borel measurable set $A \subseteq [0, 1]$.

Proof. We have to show that $\mu\left(G^{-1}([0, k])\right) = \mu([0, k])$ for $k > 0$. We have already seen that

[2]Borel sets are the sets that can be constructed from open or closed sets by repeatedly taking countable unions and intersections. Any measure defined on the Borel sets is called a Borel measure.

$G^{-1}([0,k]) = \bigcup_{n=1}^{\infty}\left[\frac{1}{k+n},\frac{1}{k}\right]$, thus we have:

$$\begin{aligned}
\mu\left(G^{-1}([0,k])\right) &= \frac{1}{\ln 2}\sum_{n=1}^{\infty}\int_{1/k+n}^{1/n}\frac{1}{1+x}\text{dx}\\
&= \frac{1}{\ln 2}\sum_{n=1}^{\infty}\left(\ln(1+\frac{1}{n}) - \ln(1+\frac{1}{k+n})\right)\\
&= \frac{1}{\ln 2}\sum_{n=1}^{\infty}\left(\ln(1+\frac{k}{n}) - \ln(1+\frac{k}{k+n})\right)\\
&= \frac{1}{\ln 2}\sum_{n=1}^{\infty}\int_{k/n+1}^{k/n}\frac{1}{1+x}\text{dx}\\
&= \mu\left([0,1]\right).
\end{aligned}$$

□

Thus according this Lemma 2.4.1, the *Gauss map* is ergodic with respect to the *Gauss measure*. Since the function logarithm is integrable in $[0,1]$, we can calculate the *Lyapunov exponent* defined in 2.3 as the following:

$$\begin{aligned}
\lambda(\gamma) &= -2\lim_{n\to n}\frac{1}{n}\left(\sum_{i=0}^{n}\ln(\gamma_i)\right)\\
&= -\frac{2}{\ln 2}\int_0^1\frac{\ln(x)}{1+x}dx\\
&= \frac{2}{\ln 2}\int_0^1\frac{\ln(x+1)}{x}dx\\
&= \frac{2}{\ln 2}\sum_{k=0}^{\infty}(-1)^k\int_0^1\frac{x^k}{k+1}dx\\
&= \frac{2}{\ln 2}\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)^2}\\
&= \frac{2}{\ln 2}\left(\frac{\pi^2}{12}\right)\\
&= \frac{\pi^2}{6\ln 2} = 2.3731..
\end{aligned}$$

Thus this value can hold for any initial point taken.

In general, in a phase space, for any ergodic transformation and invariant measure, the time average is nothing else than the average over the whole space. Thus taking an arbitrary point and then calculating the limit of the average as time goes to infinity of the *Gauss map* along the orbit means that any orbit will cover the entire space.

3. Simulation of the Gauss map in half-precision

The value of any rational initial points of the form $1/n$ where $n \in \mathbb{N}^*$ under the *Gauss map* is always zero. One may wonder what would happen if we used the floating point values of these numbers instead. Particularly, we examine the consequences of using a fixed precision whilst implementing the arithmetic operation of division in order to see what patterns will emerge. In short, we would like to see if the simulation of the *Gauss map* in half-precision will remain a chaotic discrete dynamical system.

3.1 Floating point arithmetic operations

Floating-point arithmetic is a nearly-universally used system for computer arithmetic. An important fact is that, it does not have the usual property of arithmetic operations. In addition, there is a serious difference of the finite set of floats depending on which precision we are operating, either single (32 bits), double (64 bits) or half precision. However, we can have only at most 2^L numbers using L bits meaning that there are $2^{16} = 65,536$ different half-precision floating point numbers. It may be fewer since some of them are reserved for special numbers such as Inf or NaN resulting from $0/0$. Also, these numbers need to be both positive and negative, so that there are at most 32768 positive numbers. Since the significand of the binary16 is made of 10 bits, one may think that $2^{10} = 1024$ is enough for practical purposes. However, the rapidity with which one may lose all accuracy in a computation with half-precision numbers may be a surprise. Without loss of generality, for this project, we are just focusing on positive normal numbers starting from the smallest one which is 2^{-14}.

First, let us define a new *Gauss map* noted $\hat{G}$ before any implementation.

3.1.1 Definition. Let $\hat{G} : [0, 1] \to [0, 1]$ be a map defined as follows:

$$\hat{G}(x) = \begin{cases} 0 & \text{if} \quad x = 0 \\ \frac{1}{x} \mod 1 & \text{if} \quad 0 < x \leq 1. \end{cases}$$

where the operations of division and $\mod 1$ deal with the half-precision floating points domain with round-off error. Since the significand is made of 10 bits, we are going to map $14 \times 2^{10} = 14,336$ different normal half-precision numbers in the interval $(0, 1)$ to get another of those numbers.

In any numerical simulation, accuracy is limited by errors due to round-off, discretisation and uncertainty of input data (Crofts, 2007). This following definition highlights how to estimate such error in terms of arithmetic operations.

3.1.2 Definition. A machine epsilon μ is the smallest number such that $1 + \mu > 1$. This is used to measure the effects of rounding errors made in arithmetic operations and it depends on the programming language.

If we take any rational number γ in the interval $(0, \mu)$ then eventually $G(\gamma) = 0$. This effectively limits the power of the singularity of the *Gauss map* (Corless et al., 1990).

3.1.3 Remark. Binary16 numbers has $\mu = 2^{-10} \approx 0.00097656$ in most programming language such as python, C or C++.

The main difficulty is now to figure out if this new *Gauss map* has the same properties as the previous one. More precisely whether it is still chaotic. However, this project, which explores a simple and well-understood dynamical system as implemented in half-precision, shows some surprises. Not all of the half-precision floating point numbers are representable in the system. The computer is not able to represent all real numbers since there are a finite number of them. Let us illustrate $\hat{G}$ with the following example.

3.1.4 Example. Consider the number $1/27$. This number falls between two consecutive numbers $607/16384$ and $1215/32768$ which is not true in case of the usual arithmetic operations. The binary representation of these two numbers are $1.0010111110 \times 2^{-5}$ and $1.0010111111 \times 2^{-5}$ respectively which is in fact differ from the last bit. Since the half-precision system can only represent 10 bits explicitly for the significand, thus this leaves the number $1/27$ out of the list of numbers that can be stored in the computer. This means that it cannot be represented exactly in binary16.

This Example 3.1.4 shows that the computer has chosen $607/16384$ to be represented instead of $1/27$. This number has been "correctly rounded" to the nearest half-precision number. The unit round-off error made in these half-precision numbers is $1/2048$. Thus $1/27$ is close to the lower number.

The application of this rounding in half-precision numbers induces a significant error immediately for the *Gauss map*. The number $1/27$ gives zero under G while the number $607/16384$ gives $602/607$ which has the value next to 1. however, implementing in half-precision, the number $607/16384$ gives $63/64$ under $\hat{G}$ which is still near 1 but not as close. Consequently, this single rounding in order to fit $1/27$ into the system develops a significant change of the action of the *Gauss map*, which is almost 1. This difference may still be large if we are working in high precision either single or double. The shadowing result which we give later helps to ameliorate this effect, but does not cure it.

3.2 Detection of periodic orbits cycle

Before any implementation, let us find first what are the $14,336$ half-floating point numbers in the interval $(0,1)$. To do so, the algorithm to find them is as the following:

1. Put in an array of the possibility combination of 0 and 1 for 10 bits. This is exactly all the possible significands.
2. Create an empty array of range $14,336$.
3. Set a counter.
4. Making for loops to calculate the half-precision floating point numbers.
5. Put these numbers in the empty array .

The code is simulated as follows:

Algorithm 1 Number of floats16

```
significands ← array of possible combination of 0 and 1 for 10 bits.
binary16 ← array of range(14, 336)
a ← 0
for i in range (-14,0) do
    for j in range (1024) do
        binary16[a] ← np.float16(2^i(1+sum(int(significands[j][k])/2^(k+1) for k in range(10))))
    a ← a + 1
print (binary16)
```

Now we have an array of all the binary16 numbers. We are going to map all of these under $\hat{G}$. Let us check how many of them are directly map to zero. The algorithm to find that is stated as follows:

1. Creating an array of range $14, 336$.
2. Making a foor loop of range $14, 336$.
3. Mapping each one of binary16 numbers to $\hat{G}$, and compare these values with the binary16 numbers themselves. Then take the indexes and put them inside the array defined above.

The code is implemented as follows:

Algorithm 2 half-precision under $\hat{G}$

```
array ← array of range 14,336
for i in range of (14,336) do
    y ← Ĝ(binary16[i])
    p =np.argwhere(binary16=y)
    array[i]=p[0][0]
print(np.argwhere(array!=0)
```

For finding the number of cycle of the periodicity, we are going to state again an algorithm which is the following:

1. Create an empty list.
2. Set a counter.
3. Initialize the first iteration of the orbit under the *Gauss map*.
4. Make a for loop to find the sequence of the orbit in a certain range and then put it in the empty list.
5. Make another for loop to search which value is repeating and count the number of iteration until this repetition is found.
6. Then the number of cycle is given by the subtraction of the number of the counter to the index of the number which is repeating minus one since a list is counted from zero.

Thus this code can be implemented as follows:

Algorithm 3 Number of cycles

```
procedure PERIOD(n)
    a = []
    b ← 0
    m ← Ĝ(n)
    for i in range of 2000 do
        a.append(m)
        m ← Ĝ(m)
    for i in range of the length of a do
        b ← +1

        if one of the indexed value already exists then
            return b − 1 − index(a[i])
```

The algorithm to find the transient is almost the same as this previous one. The differences are that there is no need to count the number of iteration and it has to return only the index since this is already the number that we are searching for.

Results

Recall that half-precision has higher precision for floating point values close to zero. In that way, we would like to expect having less values which do not go to zero. However, after the simulation, there are 5156 float16 numbers that are immediately mapped to 0. The smallest number that does not get mapped straight to 0 by $\hat{G}$ is 1041/1048576, which has [0; 1007, 3, 1, 1, 1, 1, 18, 1, 2] as continued fraction expansion. This was found to the 4114-th number of the binary16 array numbers. The largest one is the number $2047/2048$ which was found as the last value of the binary16 numbers.

One may wonder what will happen to all the fixed points of the *Gauss map*. Recall that they are given by the positive solution of the equation $x = \frac{1}{x} - n$ for $n \in \mathbb{N}^*$. Thus, the computer has to fit these numbers in half-precision floating points and find their own representation. After the implementation, there are only 3 fixed points under $\hat{G}$. The *Golden mean* is rounded as $633/1024$. This fixed point implemented under $\hat{G}$ is periodic with $[0; 1, 1, 1, 1, 1, 1, 1, 1]$ as continued fraction expansion. However, the rounding of the *Silver mean* in half-precision is not one of the fixed points any more. The next one is the number $155/512$ which is next to the fixed point with continued fraction $[0; 3, 3, 3, ..]$ under the exact *Gauss map*. The last one is obviously the number 0.

The number of cycle of the periodic orbits gives some remarkable results but seemingly, the shortest transient is 0 and the shortest cycle is 1. There are two 1-cycle which are only the three fixed points described above, two 2-cycle, two 3-cycle, one 6-cycle and one 18-cycle. This last one is the largest cycle for this simulation and there are no other cycles apart from these.

3.3 Orbits implemented in binary16

Since all orbits which are not mapped to zero are periodic and there are only a finite number of such orbits, $\hat{G}$ is not ultimately chaotic anymore. Two arbitrary initial points as close as possible do not have orbits moving apart, in fact they are still close or even the same. So the sensitivity of $\hat{G}$ is not valid.

As a consequence, in this new half-precision floating *Gauss map*, the Lyapunov exponent is not used. In addition, the round-off errors introduced into the calculation of the orbits is far away from the *Gauss map* G. However in such a case, one can always wonder about this limitation of floating point systems whether or not the computed solution is close to a true solution of the system of interest.

There is a technique called *Backward error analysis* which allows to evaluate the orbits of $\hat{G}$. The aim of this method is to show that despite the fact that the numerical simulation of orbits under $\hat{G}$ with round-off errors is not exactly correct, it is almost the true solution with slightly perturbed input data. Hence, there is a correlation between orbits under $\hat{G}$ and the orbits under G (Corless et al., 1990). This shows that there is a shadowing bound at the numerical values of the orbits under $\hat{G}$. Formally, there exists $\varepsilon > 0$ such that for two orbit sequences $(\gamma_i)_{i\in\mathbb{N}}$ and $(\alpha_i)_{i\in\mathbb{N}}$ relatively under $\hat{G}$ and G itself,

$$|\gamma_i - \alpha_i| < \varepsilon.$$

The following theorem shows this shadowing effect of the numerical simulation of these orbits.

3.3.1 Theorem. *(Corless et al., 1990) Let $x_0, x_1, x_2, x_3, \ldots$ be the orbit sequence under $\hat{G}$ and $y = [a_1, a_2, a_3, \ldots]$ where a_i's are the integers arising during the continuous iteration. Then the orbit of y under G approaches $orb(x_0)$. More precisely, y is close to x_0.*

Proof. The proof will proceed as follows: first, we are proceeding to give an approximation of an orbit y by a specific rational numbers. After, we are going to approximate x_k by the same rational number using a common floating-point arithmetic model. Precisely, it will be based on the fact that errors will decrease while running the *Gauss map* backwards.

Let $y_k = [a_{k+1}, a_{k+2}, \ldots]$. Truncating this continued fraction expansion of y_k at the integer a_{k+n} produces the rational numbers $\frac{p_n}{q_n} = [a_{k+1}, a_{k+2}, a_{k+3}, \ldots, a_{k+n}]$ which satisfy

$$|y_k - \frac{p_n}{q_n}| < \frac{1}{q_n^2} \quad \text{and} \quad q_n \geq F_n,$$

where F_n is the n-th Fibonnacci number (Olds, 1963), so that given $\varepsilon > 0$, we can find a number n such that $|y_k - \frac{p_n}{q_n}| < \varepsilon$.

Now, let a,b and c be three floating-point numbers satisfying $a \div b = c$. This division can happen for any numerical implementation with a precised machine epsilon μ. Thus there exists $|\delta| < \mu$ such that

$$c(1+\delta) = \frac{a}{b}.$$

Coming back to our case, if the orbit $x_0, x_1, x_2, x_3, \ldots$ has been generated in such a floating-point operating model and $G(x_{k+n}) = x_{k+n+1}$, then for each n there is a number δ_{k+n} with $|\delta_{k+n}| < \mu$ such that

$$(1+\delta_{k+n})x_{k+n} = \frac{1}{a_{k+n+1} + x_{k+n+1}},$$

where a_{k+n+1} and x_{k+n+1} are machine representable integer and floating-points respectively. Then this addition appears in the denominator is exact. Now, consider $\varepsilon_{k+n+1} = \frac{x_{k+n+1}}{a_{k+n+1}}$. Then we have

$$(1+\epsilon_{k+n+1})(1+\delta_{k+n})x_{k+n} = \frac{1}{a_{k+n+1}}.$$

Now, let

$$z_{k+m} = [a_{k+m+1}, a_{k+m+2}, a_{k+m+3}, ..., a_{k+n+1}] \quad \text{and} \quad \epsilon_{k+m} = z_{k+m} - x_{k+m} \quad \text{for } m = 0, 1, 2, ..., n$$

The aim is to estimate the error $\varepsilon_k = z_k - x_k$ since we have already seen the error for $z_k - y_k$.

So now we have

$$\begin{aligned}(1 + \delta_{k+m})x_{k+m} &= \frac{1}{a_{k+m+1} + x_{k+m+1}} \\ &= \frac{1}{a_{k+m+1} + z_{k+m+1} - \epsilon_{k+m+1}} \\ &= z_{k+m}.\frac{1}{1 - \epsilon_{k+m+1}.z_{k+m}}\end{aligned}$$

Thus

$$\begin{aligned}z_{k+m} &= (1 - \epsilon_{k+m+1}.z_{k+m})(1 + \delta_{k+m})x_{k+m} \\ &= x_{k+m} - \epsilon_{k+m+1}z_{k+m}x_{k+m} + \delta_{k+n}x_{k+m} - \delta_{k+n}\epsilon_{k+m+1}z_{k+m}x_{k+m}.\end{aligned}$$

Now we get a recurrence relation defined by

$$\begin{aligned}\epsilon_{k+m} &= z_{k+m} - x_{k+m} \\ &= \delta_{k+m}x_{k+m} - (1 + \delta_{k+m})\epsilon_{k+m+1}z_{k+m}x_{k+m}.\end{aligned}$$

This recurrence relation provides an upper bound estimated to be

$$\varepsilon_{k+m} \leq \begin{cases} 4u + \frac{1-4\mu}{2^{(n+1-m)/2}} & \textit{n-m is odd} \\ 4u + \frac{1-3\mu}{2^{(n-m)/2}} & \textit{n-m is even} \end{cases}$$

and since $n \to \infty, z_k \to y_k$, and so $|x_k - y_k| \leq 4u$.

Therefore there exists a nearby initial point y_0 whose orbit under G follows as near as possible to the computed orbit $x_0, x_1, x_2, x_3, ...$ under $\hat{G}$. □

4. Conclusion

The *Gauss map* shows a particular study of the dynamical system. The significance of the periodic orbits play an important role to determine the dynamical behaviour of the system either being chaotic in one part or not in other part. However, despite the fact that working in half precision floating points may cause some arithmetic errors, even more higher for both single and double precision, we have tried to establish that orbits under this simulation are not far away from the exact one. Some further works may determine the number of basins of attraction of the set of these periodicity cycles seen while simulating the *Gauss map* in binary16. We can also do a similar study using the single and double precision and compare how the dynamics of the *Gauss map* change.

Acknowledgements

I would like to express my first gratitude to the Almighty God for the health and strength throughout this essay.

I would like to extend my sincere and deepest thanks to my supervisor Prof. Robert Corless and my co-supervisor Prof. Nicolas Fillion for their valuable guidance and continuous support. They have been such wonderful supervisors and I am infinitely grateful.

A special profound gratitude to the following people: Prof. Barry Green(AIMS Director) Prof. J.W Sanders (Academic director), Jan Groenewald (IT manager) and my tutor Taboka Prince Chalebgwa for the opportunity and help.

I would like also to thank all AIMS family and friends for making my stay here very memorable.

Finally, big thanks to my mom (Neny), my elder brother (Zo) and my younger brother (Petit) for being supportive and for their infinite encouragement.

References

Billingsley, P. *Ergodic Theory and Information*. Wiley Series in Probability and Mathematical Statistics. Wiley, 1965. ISBN 9780471072508,0471072508.

Corless, R., Frank, G., and Graham, J. Chaos and Continued fractions. *Physica D: Nonlinear Phenomena*, 46(2):241 – 253, 1990. ISSN 0167-2789. doi: https://doi.org/10.1016/0167-2789(90)90038-Q. URL http://www.sciencedirect.com/science/article/pii/016727899090038Q.

Corless, R. M. Continued fractions and Chaos. *The American mathematical monthly*, 99(3):203–215, 1992.

Crofts, J. J. Efficient Method for Detection of Periodic Orbits in chaotic maps and flows. *arXiv preprint arXiv:0706.1940*, 2007.

Devaney, R. *An Introduction to Chaotic Dynamical Systems*. Addison-Wesley studies in nonlinearity. Addison-Wesley, 2nd ed edition, 1989.

Konsor, P. Performance Benefits for Half-precision floats, August 15, 2012. URL https://software.intel.com/en-us/articles/performance-benefits-of-half-precision-floats.

Olds, C. D. *Continued fractions* . New mathematical library. Random House, 1st edition, 1963.

Scott, T. J. Mathematics and Computer Science at Odds over Real Numbers. *SIGCSE Bull.*, 23(1): 130–139, Mar. 1991. ISSN 0097-8418. doi: 10.1145/107005.107029. URL http://doi.acm.org/10.1145/107005.107029.

Sergiy Kolyada, L. S. Topological Transitivity. URL http://www.scholarpedia.org/article/Topological_transitivity.

Van Der Zijp, J. Fast Half Float Conversions. Technical report, Working paper, 2012ftp://www.fox-toolkit. org/pub/fasthalffloatconversion. pdf, 2008.

Widž, J. From the History of Continued Fractions. In *WDS*, volume 9, pages 176–181, 2009.

Zhou, Z. Periodic Orbits on Discrete Dynamical Systems. *Computers & Mathematics with Applications*, 45(6-9):1155–1161, 2003.